

POLYGON CONVEXITY: A MINIMAL $O(n)$ TEST

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ABSTRACT. An $O(n)$ test for polygon convexity is stated and proved. It is also proved that the test is minimal in a certain exact sense.

0. INTRODUCTION

Everyone knows a convex polygon when one sees it. Yet, to deal with the notion of polygon convexity mathematically or computationally, it must be adequately described. A convex polygon can be defined, as e.g. in [6, page 5], as a succession of connected line segments which constitute the boundary of a convex set. However, in computational geometry it seems more convenient to consider a polygon as a sequence of its vertices, say (V_0, \dots, V_{n-1}) , with the edges being the segments $[V_0, V_1], \dots, [V_{n-2}, V_{n-1}], [V_{n-1}, V_0]$. Then one can say that a polygon is convex if the union of its edges coincides with the boundary of the convex hull of the set of vertices $\{V_0, \dots, V_{n-1}\}$.

One finds the following statement in [2, page 233]:

Theorem 4.3 Let the sequence of vertices, $p_1, p_2, \dots, p_n, p_{n+1} = p_1$, define an arbitrary polygon P and let P_i be the polygon defined by the sequence of vertices $p_1, p_2, \dots, p_i, p_1$. Then P is convex if and only if, for each i , $i = 3, 4, \dots, n$, polygon P_i is itself convex.

It is also said in [2] that an incremental test for polygon convexity can be based on the quoted theorem. No proof or reference to a proof of this theorem was given there. Moreover, the “if” part of the theorem is trivial: if all polygons P_3, \dots, P_n are convex, then polygon $P = P_n$ is trivially convex. Thus, such a theorem by itself would be impossible to use for an incremental test.

One might suppose that there was a typo in the quoted statement of Theorem 4.3 and there was meant to be $i = 3, 4, \dots, n - 1$ in place of $i = 3, 4, \dots, n$ (or, equivalently, $p_1, p_2, \dots, p_n, p_{n+1}, p_1$ in place of $p_1, p_2, \dots, p_n, p_{n+1} = p_1$). But then the theorem could not be true. Indeed, note that all n -gons with $n \leq 3$ are convex. Hence, if the “if” part of quoted Theorem 4.3 were true with $i = 3, 4, \dots, n - 1$ in

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place of $i = 3, 4, \dots, n$, then it would immediately follow by induction in n that all polygons whatsoever are convex!

However, it appears that the polygon convexity test suggested in [2] may be basically correct by itself, even though it is not in fact based on the quoted theorem (or proved otherwise). In this paper, we rigorously state and prove an $O(n)$ polygon convexity test, which is similar to the test suggested in [2]. Moreover, we show that our test is minimal in the sense that none of the $3(n-3)$ test conditions can be dropped if the test is to remain valid.

Under the additional condition that the n -gon is simple (that is, the only points belonging to two different edges of the n -gon are its vertices), an $O(n)$ convexity test seems to be well known [1, 2, 5] but hardly ever rigorously proved. However, no $O(n)$ simplicity tests seem to be known [2].

One may also note that the “only if” part of the quoted Theorem 4.3 turns out basically correct. Indeed, the main result in [3] states that if $\mathcal{P} = (V_0, \dots, V_{n-1})$ is a convex polygon whose vertices are all *distinct*, then the reduced polygon $\mathcal{P}^{(i)} := (V_0, \dots, V_{i-1}, V_{i+1}, \dots, V_{n-1})$ (with vertex V_i and hence edges $[V_{i-1}, V_i]$ and $[V_i, V_{i+1}]$ removed) is also convex, for each i .

In addition to such downward hereditariness of polygon convexity, it is shown in [3] that the polygon convexity property is hereditary upwards as well. Namely, if a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ with $n \geq 5$ vertices is such that all the reduced polygons $\mathcal{P}^{(i)}$ are convex, then \mathcal{P} is also convex.

Taken together, the downward and upward hereditariness of polygon convexity can be used to obtain conditions necessary and sufficient for polygon convexity. In particular, a corollary in [3] states that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ with $n \geq 5$ distinct vertices is convex if and only if all the reduced polygons $\mathcal{P}^{(i)}$ are convex. Such a test is helpful in theoretical considerations. However, it would be extremely wasteful computationally, as it takes $\Omega(n!)$ operations.

The paper is organized as follows. In Section 1, the basic definitions are given and the main results are stated: Theorem 1.5, which provides an $O(n)$ polygon convexity test; and Proposition 1.6, which shows that the test is exactly minimal in a certain sense.

In Section 2, the proofs are given. More specifically, Subsection 2.1 of Section 2 contains definitions needed in the proofs. Subsection 2.2 contains statements of lemmas and based on them proofs of the main results stated in Section 1; the proofs of all lemmas are deferred further to Subsection 2.3.

1. DEFINITIONS AND RESULTS

A *polygon* is defined in this paper as any finite sequence of points (or, interchangeably, vectors) on the Euclidean plane \mathbb{R}^2 . Let here $\mathcal{P} := (V_0, \dots, V_{n-1})$ be a polygon, which is sequence of n points; such a polygon is also called an n -gon. The points V_0, \dots, V_{n-1} are called the *vertices* of \mathcal{P} . The smallest value that one may allow for the integer n is 0, corresponding to a polygon with no vertices, that

is, to the sequence $()$ of length 0. The segments, or closed intervals,

$$[V_i, V_{i+1}] := \text{conv}\{V_i, V_{i+1}\} \quad \text{for } i \in \{0, \dots, n-1\}$$

are called the *edges* of polygon \mathcal{P} , where

$$V_n := V_0.$$

The symbol conv denotes, as usual, the convex hull [4, page 12]. Note that, if $V_i = V_{i+1}$, then the edge $[V_i, V_{i+1}]$ is a singleton set.

In general, our terminology corresponds to that in [4]. Here and in the sequel, we also use the notation

$$\overline{k, m} := \{i \in \mathbb{Z} : k \leq i \leq m\},$$

where \mathbb{Z} is the set of all integers; in particular, $\overline{k, m}$ is empty if $m < k$.

Let us define the convex hull and dimension of polygon \mathcal{P} as, respectively, the convex hull and dimension of the set of its vertices: $\text{conv } \mathcal{P} := \text{conv}\{V_0, \dots, V_{n-1}\}$ and $\dim \mathcal{P} := \dim\{V_0, \dots, V_{n-1}\} = \dim \text{conv } \mathcal{P}$.

Given the above notion of the polygon, a *convex polygon* can be defined as a polygon \mathcal{P} such that the union of the edges of \mathcal{P} coincides with the boundary $\partial \text{conv } \mathcal{P}$ of the convex hull $\text{conv } \mathcal{P}$ of \mathcal{P} ; cf. e.g. [6, page 5]. Thus, one has

Definition 1.1. A polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is convex if

$$\bigcup_{i \in \overline{0, n-1}} [V_i, V_{i+1}] = \partial \text{conv } \mathcal{P}.$$

Let us emphasize that a polygon in this paper is a sequence and therefore ordered. In particular, even if all the vertices V_0, \dots, V_{n-1} of a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ are the extreme points of the convex hull of \mathcal{P} , it does not necessarily follow that \mathcal{P} is convex. For example, consider the points $V_0 = (0, 0)$, $V_1 = (1, 0)$, $V_2 = (1, 1)$, and $V_3 = (0, 1)$. Then polygon (V_0, V_1, V_2, V_3) is convex, while polygon (V_0, V_2, V_1, V_3) is not.

In this paper, we shall be concerned foremost with strict convexity.

Definition 1.2. Let us say that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is strict if for any three distinct i, j , and k in the set $\overline{0, n-1}$, the vertices V_i, V_j , and V_k are non-collinear.

Definition 1.3. Let us say that a polygon is strictly convex if it is both strict and convex.

Remark 1.4. Any 3-gon is convex, and so, a 3-gon is strictly convex if and only if it is strict. All n -gons with $n \leq 2$ are strictly convex.

For a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$, let x_i and y_i denote the coordinates of its vertices V_i , so that

$$V_i = (x_i, y_i) \quad \text{for } i \in \overline{0, n-1}.$$

Introduce the determinants

$$(1) \quad \Delta_{\alpha,i,j} := \begin{vmatrix} 1 & x_\alpha & y_\alpha \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{vmatrix}$$

for α, i , and j in the set $\overline{0, n-1}$. Let then

$$a_i := \text{sign } \Delta_{i+1,i-1,i} = \text{sign } \Delta_{i-1,i,i+1};$$

$$b_i := \text{sign } \Delta_{0,i-1,i};$$

$$c_i := \text{sign } \Delta_{i,0,1} = \text{sign } \Delta_{0,1,i}.$$

The following theorem is the main result of this paper, which provides an $O(n)$ test of the strict convexity of a polygon.

Theorem 1.5. *An n -gon $\mathcal{P} = (V_0, \dots, V_{n-1})$ with $n \geq 4$ is strictly convex if and only if conditions*

$$(2) \quad \begin{aligned} a_i b_i &> 0, \\ a_i b_{i+1} &> 0, \\ c_i c_{i+1} &> 0 \end{aligned}$$

hold for all

$$i \in \overline{2, n-2}.$$

Proposition 1.6. *None of the $3(n-3)$ conditions in Theorem 1.5 can be omitted without (the “if” part of) Theorem 1.5 ceasing to hold.*

Thus, the test given by Theorem 1.5 is exactly minimal.

Remark 1.7. *Adding to the $3(n-3)$ conditions (2) in Theorem 1.5 the equality $b_2 = c_2$, which trivially holds for any polygon (convex or not), one can rewrite (2) as the following system of $3(n-3) + 1$ equations and one inequality:*

$$\begin{aligned} a_2 &= \dots = a_{n-2} \\ = b_2 &= \dots = b_{n-2} = b_{n-1} \\ = c_2 &= \dots = c_{n-2} = c_{n-1} \neq 0. \end{aligned}$$

2. PROOFS

2.1. More Definitions.

Definition 2.1. *Let us say that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is ordinary if its vertices are all distinct from one another: $(i \neq j \text{ \& } i \in \overline{0, n-1} \text{ \& } j \in \overline{0, n-1}) \implies V_i \neq V_j$.*

Let us say that two vertices of a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ are *adjacent* if they are the two endpoints of an edge of \mathcal{P} ; thus,

$$\{V_0, V_1\}, \{V_1, V_2\}, \dots, \{V_{n-2}, V_{n-1}\}, \{V_{n-1}, V_0\}$$

are the pairs of adjacent vertices of polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$.

Definition 2.2. Let us say that a polygon \mathcal{P} is quasi-strict if any two adjacent vertices of \mathcal{P} are not collinear with any other vertex of \mathcal{P} . More formally, a polygon \mathcal{P} is quasi-strict if, for any $i \in \overline{0, n-1}$ and any $j \in \overline{0, n-1} \setminus \{i, i \oplus 1\}$, the points V_i , $V_{i \oplus 1}$, and V_j are non-collinear, where

$$i \oplus 1 := \begin{cases} i + 1 & \text{if } i \in \overline{0, n-2}, \\ 0 & \text{if } i = n - 1. \end{cases}$$

Definition 2.3. Let us say that a polygon is quasi-strictly convex if it is both convex and quasi-strict.

Definition 2.4. Let P_0, \dots, P_m be any points on the plane, any two of which may in general coincide with each other. Let us write $P_2, \dots, P_m \llbracket [P_0, P_1]$ and say that points P_2, \dots, P_m are to one side of segment $[P_0, P_1]$ if there is a (straight) line ℓ containing $[P_0, P_1]$ and supporting to the set $\{P_0, \dots, P_m\}$; the latter, “supporting” condition means here (in accordance with [4, page 100]) that ℓ is the boundary of a closed half-plane containing the set $\{P_0, \dots, P_m\}$.

Let us write $P_2, \dots, P_m \llbracket [P_0, P_1]$ and say that points P_2, \dots, P_m are strictly to one side) of segment $[P_0, P_1]$ if $P_2, \dots, P_m \llbracket [P_0, P_1]$ and none of the points P_2, \dots, P_m is collinear with points P_0 and P_1 .

For any given $i \in \overline{0, n-1}$, let us say that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is to one side (respectively, strictly to one side) of its edge $[V_i, V_{i+1}]$ if the points of the set $\{V_j : j \in \overline{0, n-1} \setminus \{i, i \oplus 1\}\}$ are so.

Let us say that a polygon is (strictly) to-one-side if it is (strictly) to one side of every one of its edges.

2.2. Lemmas, and Proofs of Theorem 1.5 and Proposition 1.6.

Lemma 2.5. If an n -gon $\mathcal{P} = (V_0, \dots, V_{n-1})$ with $n \geq 3$ is quasi-strict, then it is ordinary.

Lemma 2.6. An n -gon with $n \geq 3$ is quasi-strictly convex if and only if it is strictly to-one-side.

Lemma 2.7. Let x_i and y_i denote the coordinates of points V_i , so that $V_i = (x_i, y_i)$ for all $i \in \overline{0, n-1}$. Then, for any choice of α, β, i , and j in $\overline{0, n-1}$,

$$V_\alpha, V_\beta \llbracket [V_i, V_j] \iff \Delta_{\alpha, i, j} \Delta_{\beta, i, j} > 0,$$

where $\Delta_{\alpha, i, j}$ are given by (1).

Lemma 2.8. For any $n \geq 4$, a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is quasi-strictly convex if and only if conditions

$$\begin{aligned} (C_1(i)) & \quad V_{i+1}, V_0 \llbracket [V_{i-1}, V_i], \\ (C_2(i)) & \quad V_{i-1}, V_0 \llbracket [V_i, V_{i+1}], \\ (C_3(i)) & \quad V_i, V_{i+1} \llbracket [V_0, V_1]. \end{aligned}$$

hold for all

$$i \in \overline{2, n-2}.$$

Lemma 2.9. *None of the $3(n-3)$ conditions $(C_\omega(i))$ ($\omega \in \{1, 2, 3\}$, $i \in \overline{2, n-2}$) in Lemma 2.8 can be omitted without (the “if” part of) Lemma 2.8 ceasing to hold.*

Lemma 2.10. *If a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is quasi-strictly convex, then it remains so after the elimination of any one (and hence any number) of its vertices; in particular, then the polygon $\mathcal{P}_{n-1} := (V_0, \dots, V_{n-2})$ is quasi-strictly convex.*

(Cf. the main result in [3].)

Lemma 2.11. *A convex polygon is strict if and only if it is quasi-strict.*

Proof of Theorem 1.5. This follows immediately from Lemma 2.11, Lemma 2.8, and Lemma 2.7. \square

Proof of Proposition 1.6. This follows immediately from Lemma 2.11, Lemma 2.9, and Lemma 2.7. \square

2.3. Proofs of the Lemmas.

Proof of Lemma 2.5. Indeed, if $V_i = V_j$ while $0 \leq i < j \leq n-1$, then (recalling Definition 2.2) one sees that $i \oplus 1 = i+1$ and the points V_i , $V_{i \oplus 1}$, and V_j are collinear.

If at that $j \neq i+1$, then $j \in \overline{0, n-1} \setminus \{i, i \oplus 1\}$, so that polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is not quasi-strict. Next, the set $\overline{0, n-1} \setminus \{i, i \oplus 1\}$ is non-empty (because $n \geq 3$), so that there exists some $k \in \overline{0, n-1} \setminus \{i, i \oplus 1\}$. If now $j = i+1$, then the three points V_i , $V_{i \oplus 1} = V_{i+1} = V_j = V_i$, and V_k are trivially collinear, so that again one concludes that \mathcal{P} is not quasi-strict. \square

Proof of Lemma 2.6. Observe first that a polygon is strictly to-one-side if and only if it is quasi-strict and to-one-side. (This follows immediately from Definitions 2.4 and 2.2.) Also, by Lemma 2.5, every quasi-strict n -gon with $n \geq 3$ is ordinary. On the other hand, it was shown in [3] that an ordinary polygon is convex if and only if it is to-one-side. Now Lemma 2.6 follows. \square

Proof of Lemma 2.7. Take any α, β, i, j in the set $\overline{0, n-1}$. By Definition 2.4, one has $V_\alpha, V_\beta \ll [V_i, V_j]$ if and only if $V_j \neq V_i$ and there exists some vector $\vec{n} = (a, b) \in \mathbb{R}^2$ such that

$$\vec{n} \cdot \overrightarrow{V_i V_j} = 0 < \vec{n} \cdot \overrightarrow{V_i V_\gamma} \quad \text{for } \gamma \in \{\alpha, \beta\}.$$

Since

$$\Delta_{\alpha, i, j} = \begin{vmatrix} 1 & x_\alpha - x_i & y_\alpha - y_i \\ 1 & 0 & 0 \\ 1 & x_j - x_i & y_j - y_i \end{vmatrix},$$

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one may replace without loss of generality (w.l.o.g.) the points $V_\alpha, V_\beta, V_i, V_j$ by $V_\alpha - V_i, V_\beta - V_i, V_i - V_i = (0, 0), V_j - V_i$, respectively. Hence, w.l.o.g.

$$V_i = (0, 0).$$

Then the condition $V_\alpha, V_\beta \ll [V_i, V_j]$ can be rewritten as follows:

$$(x_j, y_j) \neq (0, 0) \quad \text{and} \quad ax_j + by_j = 0 < ax_\gamma + by_\gamma \quad \text{for } \gamma \in \{\alpha, \beta\}.$$

W.l.o.g., $y_j \neq 0$. Then condition $ax_j + by_j = 0$ is equivalent to $b = -\frac{x_j}{y_j}a$, so that the inequality $0 < ax_\gamma + by_\gamma$ can be rewritten as $\frac{a}{y_j}(x_\gamma y_j - x_j y_\gamma) > 0$, or as $\frac{a}{y_j}\Delta_{\gamma, i, j} < 0$ (where $\gamma \in \{\alpha, \beta\}$); in particular, it follows that $a \neq 0$.

We see that the condition $V_\alpha, V_\beta \ll [V_i, V_j]$ implies

$$\Delta_{\alpha, i, j}\Delta_{\beta, i, j} = \left(\frac{y_j}{a}\right)^2 \left(\frac{a}{y_j}\Delta_{\alpha, i, j}\right) \left(\frac{a}{y_j}\Delta_{\beta, i, j}\right) > 0.$$

This proves the “ \implies ” part of Lemma 2.7.

To prove the “ \impliedby ” part, let $\vec{n} := \varepsilon(-y_j, x_j)$, where $\varepsilon := \text{sign } \Delta_{\alpha, i, j}$. Then the condition $\Delta_{\alpha, i, j}\Delta_{\beta, i, j} > 0$ implies that $\varepsilon = \text{sign } \Delta_{\beta, i, j}$. Also, one has $\vec{n} \cdot \vec{V_i V_j} = 0$, while

$$\vec{n} \cdot \vec{V_i V_\gamma} = \varepsilon(x_j y_\gamma - y_j x_\gamma) = \varepsilon \Delta_{\gamma, i, j} = |\Delta_{\gamma, i, j}| > 0$$

for $\gamma \in \{\alpha, \beta\}$, so that the condition $V_\alpha, V_\beta \ll [V_i, V_j]$ takes place. \square

Proof of Lemma 2.8. “Only if” This part of Lemma 2.8 follows immediately from Lemma 2.6.

“If” Assume that indeed conditions $(C_1(i))$, $(C_2(i))$, and $(C_3(i))$ hold for all $i \in \overline{2, n-2}$. To prove the “if” part of Lemma 2.8, it suffices to show that, for all $k \in \overline{3, n}$, the polygon $\mathcal{P}_k := (V_0, \dots, V_{k-1})$ is quasi-strictly convex. We shall do this by induction in k .

For $k = 3$, the polygon $\mathcal{P}_k = \mathcal{P}_3 = (V_0, V_1, V_2)$ is quasi-strict, in view of condition $(C_1(2))$ and Definitions 2.4 and 2.2. Therefore, \mathcal{P}_k is quasi-strictly convex for $k = 3$.

Suppose now that

$$k \in \overline{3, n-1}$$

and \mathcal{P}_k is quasi-strictly convex. We have then to verify that polygon $\mathcal{P}_{k+1} = (V_0, \dots, V_k)$ is quasi-strictly convex.

Since $k \in \overline{3, n-1}$, one has $k-1 \in \overline{2, n-2}$. Hence, condition $(C_2(k-1))$ holds, and it implies that the points V_0, V_{k-1} , and V_k are non-collinear. Therefore, w.l.o.g.

$$V_k = (0, 0), \quad V_0 = (1, 0), \quad V_{k-1} = (0, 1).$$

Let also

$$V_{k-2} = (u, v) \quad \text{and} \quad V_1 = (x, y),$$

for some real x, y, u , and v . Finally, take any

$$i \in \overline{1, k-2}$$

and let

$$V_i = (\lambda, \mu),$$

for some real λ and μ .

Since $k-1 \in \overline{2, n-2}$, conditions $(C_1(k-1))$, $(C_2(k-1))$, $(C_3(k-1))$ hold. In view of Lemma 2.7, these three conditions yield respectively that

$$(3) \quad u(u+v-1) > 0,$$

$$(4) \quad u > 0,$$

$$(5) \quad (x+y-1)y > 0.$$

Because polygon $\mathcal{P}_k = (V_0, \dots, V_{k-1})$ is assumed to be quasi-strictly convex, it follows by Lemma 2.6 that \mathcal{P}_k is strictly to one side of every one of its edges,

$$[V_0, V_1], \dots, [V_{k-2}, V_{k-1}], [V_{k-1}, V_0].$$

In particular, one has $V_{k-2}, V_1 \ll [V_0, V_{k-1}]$ (because the condition $k \in \overline{3, n-1}$ implies that $k-2 \neq 0$ and $1 \neq k-1$). In view of Lemma 2.7, this yields

$$(1-u-v)(1-x-y) > 0.$$

Now it follows from (3)–(5) that

$$(6) \quad u > 0,$$

$$(7) \quad u+v-1 > 0,$$

$$(8) \quad x+y-1 > 0,$$

$$(9) \quad y > 0.$$

Moreover, the quasi-strict convexity of polygon \mathcal{P}_k and Lemma 2.6 imply relations

$$V_i, V_1 \ll [V_0, V_{k-1}], \quad V_i, V_{k-1} \ll [V_0, V_1], \quad V_i, V_0 \ll [V_{k-2}, V_{k-1}],$$

which in turn yield

$$(1-\mu-\lambda)(1-x-y) > 0,$$

$$((1-\lambda)y + \mu(x-1))(x+y-1) \geq 0,$$

$$((1-\mu)u + \lambda(v-1))(u+v-1) \geq 0,$$

respectively (the last two inequalities are in fact strict except for the cases $i = 1$ for the former and $i = k-2$ for the latter). In view of (8) and (7), these three inequalities imply

$$(10) \quad \lambda + \mu - 1 > 0,$$

$$(11) \quad (1-\lambda)y + \mu(x-1) \geq 0,$$

$$(12) \quad (1-\mu)u + \lambda(v-1) \geq 0.$$

Next, (6) and (10) imply $u(\lambda + \mu - 1) > 0$. Adding this inequality to (12), one has $\lambda(u+v-1) > 0$. Now (7) yields

$$(13) \quad \lambda > 0.$$

In view of Lemma 2.7, this is equivalent to $V_i, V_0 \ll [V_{k-1}, V_k]$, for all $i \in \overline{1, k-2}$. That is,

$$(14) \quad V_0, \dots, V_{k-2} \ll [V_{k-1}, V_k].$$

Similarly, (9) and (10) imply $y(\lambda + \mu - 1) > 0$. Adding this inequality to (11), one has $\mu(x + y - 1) > 0$. Now (8) yields

$$(15) \quad \mu > 0,$$

which is equivalent to $V_i, V_{k-1} \ll [V_k, V_0]$, for all $i \in \overline{1, k-2}$. That is,

$$(16) \quad V_1, \dots, V_{k-1} \ll [V_k, V_0].$$

Also, since condition $(C_3(i))$ was assumed to hold for all $i \in \overline{2, n-2}$, one has $V_2, \dots, V_{n-1} \ll [V_0, V_1]$. Hence and because $k \in \overline{3, n-1}$,

$$(17) \quad V_2, \dots, V_k \ll [V_0, V_1].$$

Suppose that the following sublemma of Lemma 2.8 is true (we shall prove the sublemma after the proof of Lemma 2.8 is completed).

Sublemma 2.12. *For all $i \in \overline{1, k-2}$, one has $V_k, V_0 \ll [V_i, V_{i+1}]$.*

Let us now complete the proof of Lemma 2.8. Since polygon \mathcal{P}_k is assumed to be quasi-strictly convex, Lemma 2.6 implies that, for all $i \in \overline{1, k-2}$,

$$V_0, \dots, V_{i-1}, V_{i+2}, \dots, V_{k-1} \ll [V_i, V_{i+1}],$$

and so, by Sublemma 2.12,

$$(18) \quad V_0, \dots, V_{i-1}, V_{i+2}, \dots, V_{k-1}, V_k \ll [V_i, V_{i+1}] \quad \text{for all } i \in \overline{1, k-2}.$$

Relations (14), (16), (17), and (18) taken together mean that polygon \mathcal{P}_k is strictly to one side of every one of its edges,

$$[V_0, V_1], \dots, [V_{k-1}, V_k], [V_k, V_0].$$

Hence, by Lemma 2.6, polygon \mathcal{P}_{k+1} is quasi-strictly convex. Thus, the induction step is verified. \square

Proof of Sublemma 2.12. Take any $i \in \overline{1, k-2}$ and let

$$V_{i+1} = (a, b),$$

for some real a and b . We need to show that $V_k, V_0 \ll [V_i, V_{i+1}]$. If $i = k-2$, then condition $V_k, V_0 \ll [V_i, V_{i+1}]$ coincides with condition $(C_1(k-1))$. Hence, w.l.o.g.

$$i \in \overline{1, k-3},$$

so that $\{k-1, 0\} \cap \{i, i+1\} = \emptyset$. Therefore and because polygon $\mathcal{P}_k = (V_0, \dots, V_{k-1})$ was assumed to be quasi-strictly convex, Lemma 2.6 yields $V_{k-1}, V_0 \ll [V_i, V_{i+1}]$. By Lemma 2.7, the latter relation can be rewritten as

$$(19) \quad (\lambda q - \mu p + p)(\lambda q - \mu p - q) > 0,$$

where

$$p := a - \lambda \quad \text{and} \quad q := b - \mu.$$

On the other hand, relation $V_k, V_0 \ll [V_i, V_{i+1}]$ (which is to be proved here) can be rewritten as

$$(20) \quad (\lambda q - \mu p)(\lambda q - \mu p - q) > 0.$$

Consider separately the following three cases, depending on whether $\lambda q - \mu p$ is zero, positive, or negative.

Case 1: $\lambda q - \mu p = 0$. Then (13) and (15) yield $pq \geq 0$, while (19) implies $pq < 0$, which is a contradiction.

Case 2: $\lambda q - \mu p > 0$. Here, if (20) failed to hold, one would have

$$(21) \quad \lambda q - \mu p - q \leq 0$$

and hence also

$$(22) \quad q > 0.$$

Now (19) would imply

$$(23) \quad \lambda q - \mu p + p < 0$$

and hence also

$$(24) \quad p < 0.$$

Next, (22) and (10) would yield $(-q)(\lambda + \mu - 1) < 0$. Adding the latter inequality to (21), one would have $(-\mu)(p + q) < 0$, which would result (in view of (15)) in

$$(25) \quad p + q > 0.$$

On the other hand, (24) and (10) would yield $p(\lambda + \mu - 1) < 0$. Adding the latter inequality to (23), one would have $\lambda(p + q) < 0$ and then, in view of (13), $p + q < 0$, which would contradict (25).

Case 3: $\lambda q - \mu p < 0$. This case is quite similar to Case 2: just switch the direction of all inequalities obtained in the consideration of Case 2. \square

Proof of Lemma 2.9. The proof is based on

Sublemma 2.13. *Let $\mathcal{P}_k := (V_0, \dots, V_{k-1})$ be any quasi-strict k -gon with $k \geq 3$. Then*

- (i): *there exists a point V_k such that the $(k+1)$ -gon $\mathcal{P}_{k+1} := (V_0, \dots, V_{k-1}, V_k)$ is quasi-strict and satisfies the condition*

$$(C_1(k-1)) \ \& \ (C_2(k-1)) \ \& \ (C_3(k-1)) \ ;$$

- (ii): *there exists a point V_k such that the $(k+1)$ -gon $\mathcal{P}_{k+1} := (V_0, \dots, V_{k-1}, V_k)$ is quasi-strict and satisfies the condition*

$$(C_1(k-1)) \ \& \ (C_2(k-1)) \ \& \ (\neg C_3(k-1)) \ ;$$

here and in what follows, \neg is the usual negation symbol, so that $(\neg C_3(k-1))$ means that $(C_3(k-1))$ does not hold;

(iii): there exists a point V_k such that the $(k+1)$ -gon $\mathcal{P}_{k+1} := (V_0, \dots, V_{k-1}, V_k)$ is quasi-strict and satisfies the condition

$$(C_1(k-1)) \ \& \ (\neg C_2(k-1)) \ \& \ (C_3(k-1)) \ ;$$

(iv): there exists a point V_k such that the $(k+1)$ -gon $\mathcal{P}_{k+1} := (V_0, \dots, V_{k-1}, V_k)$ is quasi-strict and satisfies the condition

$$(\neg C_1(k-1)) \ \& \ (C_2(k-1)) \ \& \ (C_3(k-1)) \ .$$

We shall prove this sublemma later. Now, let us complete the proof of Lemma 2.9. For each $\omega \in \{1, 2, 3\}$ and each set $J \subseteq \overline{2, n-2}$, introduce the condition

$$(C_\omega(J)) := (\forall i \in J \ (C_\omega(i))),$$

which is the conjunction of conditions $(C_\omega(i))$ over all $i \in J$.

Consider the following statement, for $n \geq 3$:

$$\begin{aligned} & \text{for every } i \in \overline{2, n-2} \text{ there exists a quasi-strict } n\text{-gon } \mathcal{P}_n := \\ (M_3(n)) \quad & (V_0, \dots, V_{n-1}) \text{ satisfying the condition} \\ & (C_1(\overline{2, n-2})) \ \& \ (C_2(\overline{2, n-2})) \ \& \ (C_3(\overline{2, n-2} \setminus \{i\})) \ \& \ (\neg C_3(i)). \end{aligned}$$

We shall prove statement $(M_3(n))$ by induction in n . If $n = 3$, then $\overline{2, n-2} = \emptyset$, so that $(M_3(n))$ trivially holds.

Suppose next that statement $(M_3(n))$ holds for some $n = k$, where $k \geq 3$. We have to verify that then statement $(M_3(n))$ holds for $n = k+1$. For $n = k+1$ and $i \in \overline{2, n-2}$, only two cases are possible: $i \in \overline{2, k-2}$ or $i = k-1$. Let us consider these two cases separately.

Case 1: $i \in \overline{2, k-2}$. In this case, by induction, there exists a quasi-strict k -gon $\mathcal{P}_k := (V_0, \dots, V_{k-1})$ satisfying the condition

$$(C_1(\overline{2, k-2})) \ \& \ (C_2(\overline{2, k-2})) \ \& \ (C_3(\overline{2, k-2} \setminus \{i\})) \ \& \ (\neg C_3(i)).$$

By part (i) of Sublemma 2.13, there exists a point V_k such that the $(k+1)$ -gon $\mathcal{P}_{k+1} := (V_0, \dots, V_{k-1}, V_k)$ is quasi-strict and satisfies the condition

$$(C_1(k-1)) \ \& \ (C_2(k-1)) \ \& \ (C_3(k-1)) \ .$$

It follows that \mathcal{P}_{k+1} satisfies the condition

$$(26) \quad (C_1(\overline{2, k-1})) \ \& \ (C_2(\overline{2, k-1})) \ \& \ (C_3(\overline{2, k-1} \setminus \{i\})) \ \& \ (\neg C_3(i)).$$

Case 2: $i = k-1$. For every $k \geq 3$, there is a quasi-strict k -gon $\mathcal{P}_k := (V_0, \dots, V_{k-1})$ satisfying the condition

$$(C_1(\overline{2, k-2})) \ \& \ (C_2(\overline{2, k-2})) \ \& \ (C_3(\overline{2, k-2})).$$

(This follows by induction using part (i) of Sublemma 2.13.) Let \mathcal{P}_k be such a k -gon. By part (ii) of Sublemma 2.13, there exists a point V_k such that the $(k+1)$ -gon $\mathcal{P}_{k+1} := (V_0, \dots, V_{k-1}, V_k)$ is quasi-strict and satisfies the condition

$$(C_1(k-1)) \ \& \ (C_2(k-1)) \ \& \ (\neg C_3(k-1)) \ ,$$

so that (26) again holds—with $i = k-1$.

Thus, statement $(M_3(n))$ takes place for $n = k + 1$, and hence for all $n \geq 3$. This implies that none of the $n - 3$ conditions $(C_3(i))$ with $i \in \overline{2, n - 2}$ in Lemma 2.8 can be omitted (because, by Lemma 2.6, all of the conditions $(C_3(i))$ with $i \in \overline{2, n - 2}$ are necessary for polygon \mathcal{P} to be quasi-strictly convex).

Similarly (but using parts (iii) and (iv) of Sublemma 2.13 rather than part (ii) of it), one can show that none of the conditions $(C_2(i))$ or $(C_1(i))$ (with $i \in \overline{2, n - 2}$) in Lemma 2.8 can be omitted. \square

Proof of Sublemma 2.13. Since polygon $\mathcal{P}_k = (V_0, \dots, V_{k-1})$ is quasi-strict and $k \geq 3$, the points V_0, V_1 , and V_{k-1} are non-collinear, so that w.l.o.g.

$$V_0 = (0, 0), \quad V_1 = (1, 0), \quad V_{k-1} = (0, 1).$$

Let also

$$V_{k-2} = (x, y), \quad V_k = (u, v)$$

for some real x, y, u, v . At that, the values of x and y are given to us, while the values of u and v we are free to choose. Note that $x \neq 0$, because polygon $\mathcal{P}_k = (V_0, \dots, V_{k-1})$ is quasi-strict and hence the points V_0, V_{k-2} , and V_{k-1} are non-collinear.

Now, in view of Lemma 2.7, conditions $(C_1(k - 1))$, $(C_2(k - 1))$, $(C_3(k - 1))$ can be rewritten, respectively, as

$$(27) \quad (x - u + uy - vx)x > 0,$$

$$(28) \quad (x - u + uy - vx)(-u) > 0,$$

$$(29) \quad v > 0.$$

Now we are ready to prove parts (i)–(iv) of Sublemma 2.13.

(i): For any given values of $x \neq 0$ and y , let

$$u := -\varepsilon x, \quad v := \varepsilon + \varepsilon^2$$

for $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := 0.1/(1 + |y|)$. Then

$$|-u + uy - vx| \leq |u|(1 + |y|) + |v||x| \leq 0.1|x| + 0.11|x| < |x|,$$

whence $\text{sign}(x - u + uy - vx) = \text{sign } x$, and so, all of the conditions (27), (28), (29) hold; that is, conditions $(C_1(k - 1))$, $(C_2(k - 1))$, $(C_3(k - 1))$ hold for all V_k lying on the parabolic arc

$$P := \{(-\varepsilon x, \varepsilon + \varepsilon^2) : \varepsilon \in (0, \varepsilon_0)\}.$$

On the other hand, by Lemma 2.5, the k -gon $\mathcal{P}_k = (V_0, \dots, V_{k-1})$ is ordinary. Hence, for the $(k + 1)$ -gon $\mathcal{P}_{k+1} = (V_0, \dots, V_{k-1}, V_k)$ not to be quasi-strict, the vertex V_k must lie on the line through points V_i and V_j for some i and j such that $0 \leq i < j \leq k - 1$. But any one of these (finitely many) lines can have at most two points in common with the parabolic arc P ; hence, the union of all such lines through points V_i and V_j cannot cover the infinite set P . This means that one can find a point V_k in P such that the $(k + 1)$ -gon $\mathcal{P}_{k+1} = (V_0, \dots, V_{k-1}, V_k)$ is quasi-strict and satisfies conditions $(C_1(k - 1))$, $(C_2(k - 1))$, $(C_3(k - 1))$.

(ii): Similarly to the above, it can be seen that one can choose V_k on the parabolic arc

$$\{(-\varepsilon x, -\varepsilon - \varepsilon^2): \varepsilon \in (0, \varepsilon_0)\}$$

so that the $(k+1)$ -gon \mathcal{P}_{k+1} is quasi-strict and satisfies conditions $(C_1(k-1))$ and $(C_2(k-1))$ but not $(C_3(k-1))$.

(iii): Similarly, one can choose V_k on the parabolic arc

$$\{(\varepsilon x, \varepsilon + \varepsilon^2): \varepsilon \in (0, \varepsilon_0)\}$$

so that the $(k+1)$ -gon \mathcal{P}_{k+1} is quasi-strict and satisfies conditions $(C_1(k-1))$ and $(C_3(k-1))$ but not $(C_2(k-1))$.

(iv): Similarly, one can choose V_k on the parabolic arc

$$\{((1+\varepsilon)x, (1+\varepsilon)|y| + \varepsilon^2): \varepsilon > 0\}$$

so that the $(k+1)$ -gon \mathcal{P}_{k+1} is quasi-strict and satisfies conditions $(C_2(k-1))$ and $(C_3(k-1))$ but not $(C_1(k-1))$. (Note that the conditions $u = (1+\varepsilon)x$, $v = (1+\varepsilon)|y| + \varepsilon^2$, and $\varepsilon > 0$ imply

$$\frac{1}{x}(x - u + uy - vx) = -(\varepsilon + \varepsilon^2 + (1+\varepsilon)(|y| - y)) < 0.)$$

□

Proof of Lemma 2.10. Since one can do a cyclic permutation, it suffices to show that, if a polygon $\mathcal{P}_n = (V_0, \dots, V_{n-1})$ is quasi-strictly convex, then $\mathcal{P}_{n-1} = (V_0, \dots, V_{n-2})$ is so.

Observe that, if $n \leq 4$ and polygon \mathcal{P}_n is quasi-strict, then \mathcal{P}_{n-1} is quasi-strict. (Indeed, if $i \in \overline{0, n-3}$ and $j \in \overline{0, n-2} \setminus \{i, i+1\}$, then the points V_i , V_{i+1} , and V_j are non-collinear, because polygon $\mathcal{P}_n = (V_0, \dots, V_{n-1})$ is quasi-strict. If $j \in \overline{0, n-2} \setminus \{0, n-2\}$ and $n \leq 4$, then one must have $n = 4$ and $j = 1$, whence the points V_{n-2} , V_0 , and $V_j = V_1$ are non-collinear, because polygon $\mathcal{P}_n = (V_0, \dots, V_{n-1})$ is quasi-strict.

Moreover, for $n \leq 4$ the $(n-1)$ -gon \mathcal{P}_{n-1} is always convex. Being also quasi-strict, \mathcal{P}_{n-1} is then quasi-strictly convex.

Assume now that $n \geq 5$ and polygon \mathcal{P}_n is quasi-strictly convex. Then, by Lemma 2.8, one has $(C_1(i))$, $(C_2(i))$, and $(C_3(i))$ for all $i \in \overline{2, n-2}$ and hence for all $i \in \overline{2, (n-1)-2}$. Therefore, Lemma 2.8 implies that \mathcal{P}_{n-1} is quasi-strictly convex. □

Proof of Lemma 2.11. “Only if” The “only if” part of Lemma 2.11 is trivial.

“If” This part is proved by induction in n . The case $n \leq 2$ is trivial, because then there are no three distinct i , j , and k in the set $\overline{0, n-1}$.

Let then $n \geq 3$. Assume that the vertices V_i , V_j , and V_k of polygon $\mathcal{P}_n := (V_0, \dots, V_{n-1})$ are collinear for some distinct i , j , and k in $\overline{0, n-1}$. W.l.o.g., $0 = i < j < k \leq n-1$. Moreover, then $k \neq n-1$, because vertices V_{n-1} and $V_n = V_0$ of polygon \mathcal{P}_n are adjacent to each other. Hence, $0 = i < j < k \leq n-2$,

so that the points V_i , V_j , and V_k are vertices of polygon $\mathcal{P}_{n-1} := (V_0, \dots, V_{n-2})$. But, by Lemma 2.10, polygon \mathcal{P}_{n-1} is quasi-strictly convex. Hence, by induction, V_i , V_j , and V_k are non-collinear. \square

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